MAXIMIN TESTING OF SATELLITE STABILIZATION

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A system for accuracy testing of algorithms now in use for the active magnetic stabilization of satellite attitude is considered. The problem on accuracy testing of stabilization of equilibrium positions for a control stochastic system is formulated for the case when some parametric and time-varying perturbations are known. Some conditions for finding the optimal counterstrategy and the best result of testing are given. An example of quality testing of the system for the active magnetic stabilization of pitching oscillations of an artificial Earth satellite is examined.

1. Introduction. At present, orientation systems are widely used to control the attitude of small satellites. Measuring sensors and operating actuators of these systems interact with the Earth's magnetic field. Such systems offer certain advantages: high reliability, small weight, ease of manufacture, and relatively low cost (this is very important for small satellites).

The development of an orientation system includes testing its quality in addition to the synthesis of control algorithms. The use of computerized land-based testing devices is one of the possible methods for solving this problem. These devices significantly simplify the processes of production and testing of satellites. Figure 1 illustrates the functional block diagram of one of the possible systems for testing the algorithms of magnetic stabilization.

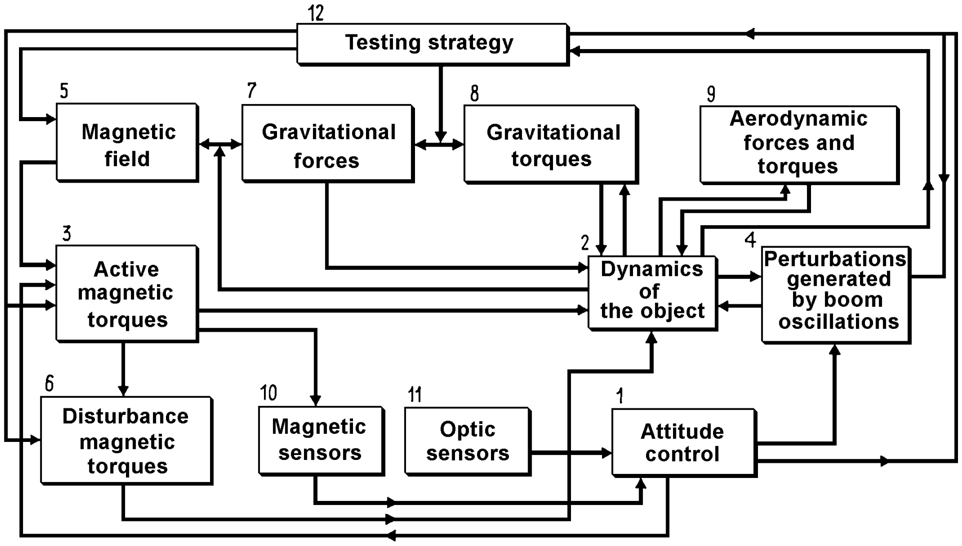

Fig. 1

The algorithms intended to control the satellite attitude are represented in Figure 1 as Block 1 to be tested. The testing system regards these algorithms as a "black box" with known inputs and outputs. This system forms outer and inner conditions under which the tested block operates. Block 2 is a basic one; it is supposed to model the dynamics of the body. This block performs the integration of equations for the center-of-mass matrix of the satellite as well as the integration of equations for the motion about this center under the action of control and disturbance torques. It is possible to include some additional elements into the model: booms for gravitational stabilization, panels of solar batteries, etc. The satellite is controlled by magnetic coils. Block 1

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generates controlling signals for the block of forming the controlling magnetic torques (Block 3). In addition, Block 3 uses the information on strength of the Earth's magnetic field whose model is realized in Block 5. This block may have several levels varying in complexity of description of the Earth's magnetic field. The disturbance magnetic torque acting on the satellite is formed in Block 6. The gravitational forces and torques acting on the satellite are modeled in Blocks 7 and 8. The gravity field is assumed central when the gravitational torques are calculated. Some models of various levels of complexity for the description of the Earth's gravitational potential are used to model the gravitational forces (Block 7). Block 9 is intended to model the behavior of the atmosphere. Blocks 5, 7, 8, and 9 altogether form the outer conditions that influence the motion of the body under control.

The inner conditions affecting the body are formed by measurement errors, errors of operating actuators, intrinsic magnetic torques as well as by a number of malfunctions and emergency situations arising during the process of operation of the satellite. It is assumed that the tested control system may use the readings of magnetic and optic sensors. The errors of the sensors are modeled in such a way that their additive, multiplicative, and random components are taken into account. The statistical description of purely random components of measurement errors as well as the description of the functional set for nonrandom components are used in the following two cases: when operation of the sensors is modeled in Blocks 10 and 11 and when Block 12 (controlling the blocks listed above and realizing the strategy of testing) is active during the process of determination of the worst perturbations. During the test, Block 12 can change the value of any parameter influencing the motion of the system, interrupt the analysis of the motion of the satellite and restart it with new initial data, etc. This block has the following input parameters: descriptions of sets of permissible ranges in specification of the designed orbit as well as descriptions of ranges of the mass and geometric parameters of the satellite and its elements (booms, solar batteries, etc.). Descriptions of sets defining the structure and parameters of each component of measurement errors of the available sensors are also specified. In addition, a description of atmospheric changes and data on gravitational and magnetic perturbations are delivered at the input of Block 12.

2. Formulation of the problem of testing. The testing procedure realized in Block 12 is based on obtaining the reliable data that characterize the stabilization accuracy for the case when the parametric and time-varying perturbations preventing the system from being stabilized are the worst possible. Such an approach was proposed in [1], where formulations of maximin problems provide the basis for the testing strategy. In this paper, this approach is extended to the case when a statistical description is known for a certain part of perturbations, which may be represented, therefore, by steady stochastic processes with a given spectral density. As is known, such perturbations can be modeled by perturbations at the output of a linear stochastic system whose input actions may be regarded as white noise with a given intensity [2]; without loss of generality, hence, we may assume that the random perturbations acting on the system are modeled by the vector white noise with a known intensity. As shown below, in this case the testing procedure [1] leads to game problems for dispersion relations of a stochastic system (these relations are expressed in terms of deviations from the required motion of the system).

Suppose the equations

$$\dot{y} = \tilde{f}(y, w, v, \xi)$$

describe the motion of a control system. Here w is the vector of control signals: $w(\cdot) \in \tilde{W} = \{w(\cdot) \in L_2^s \mid w(t) \in \Omega \subset R^s\}$; v is the vector of perturbations influencing the stabilization accuracy adversely: $v(t) \in V = \{v(\cdot) \in L_2^l \mid v(t) \in \mathcal{V} \subset R^l\}$; ξ is the vector of random perturbations modeled by a steady stochastic process.

Let us consider the system of equations

$$\dot{x} = f(x, v, u, \xi) \tag{1}$$

given in deviations $x = y - \tilde{y}$ from the desired motion $\tilde{y}(t)$ belonging to the preassigned trajectory set $Y = \{\tilde{y}(t) \in Y^0 \subset R^n,\ t_0 \leqslant t \leqslant t_k\}$. Here $u = w - \tilde{w}$ are the unutilized resources in control for solving the stabilization problem. Usually, the control signals are formed as functions of the time t and of the deviation estimates $\tilde{x}$: $u = u(t, \tilde{x})$. The accuracy of robust stabilization is defined by the functional class

$$J = M\left[\int_{t_0}^{t_k} (x^\top G x + u^\top N u)\, dt + x^\top(t_k) S x(t_k)\right] \tag{2}$$

where the matrices G and S are constant, symmetric, and positive semidefinite ($G^\top = G \geqslant 0$, $S^\top = S \geqslant 0$) and the matrix N is positive definite ($N^\top = N > 0$). It is assumed that t_0 is a fixed instant of time and t_k is the instant when the given manifold is first reached (in particular, t_k may be fixed).

The best accuracy J_0^0 (or the best result in terms of accuracy of robust stabilization) is defined as

$$J_0^0 = \sup_{v \in V} \inf_{u(\cdot) \in U} J \tag{3}$$

The best accuracy concept coincides with that of minimax stabilization if the upper and lower bounds are attainable and a saddle point $(v^0,\, u^0)$ is in existence:

$$J_0^0(u^0, v^0) = \max_{v \in V} \min_{u \in U} J(u,v) = \min_{u \in U} \max_{v \in V} J(u,v) \tag{4}$$

The vector function v depends on the deviations and control values: $v = v(t,x,u)$. Thus, there exists a certain discrimination of control; in accordance with the Krasovskii theorem [3], differential game (3) has the saddle point (4) (the instant t_k is fixed).

The testing procedure may be decomposed into the following three stages.

The first stage (preliminary). The best accuracy J_0^0 (a value of game (4)) and the optimal strategy $v^0(\cdot)$ of testing are determined.

The second stage (basic). The process of testing is simulated with the known strategy $v^0(\cdot)$.

The third stage (final). Two comparisons are performed: a) the best accuracy J_0^0 with the point and (or) interval estimates of the functional $J(u^r, v^0)$ obtained in accordance with the results of simulation; b) the stabilization strategy $u^r(\cdot)$ with the best control $u^0(\cdot)$ obtained at the first stage.

3. Determination of the optimal counterstrategy and a value of the game. Let us consider the first stage of the testing procedure: obtaining the best accuracy and forming the optimal strategy. To accomplish this, as was stated above, maximin problem (4) should be solved.

The following system of equations in deviations at the first approximation is introduced:

$$\dot{x} = A(t,p)\,x + B(t,p)\,u + C(t,p)\,q + D(t,p)\,\xi \tag{5}$$

Here ξ is the vector white noise with a known intensity matrix $F(t) \geqslant 0$ ($M[\xi(t)] \equiv 0$, $M[\xi(t)\,\xi^{\top}(t')] = F(t)\,\delta(t-t')$, $F(t)$ is a positive semidefinite symmetric matrix, and M is the expectation operator); $M[x(0)] = e_0$ is the mean value of the deviations $x(0)$ and $M\big[\mathring{x}\,(0)\;\mathring{x}^{\top}\,(0)\big] = P_0$ is the covariance matrix; p and q are the components of the vector $v = \{p,q\}$ of perturbations being varied. The subvector p describes the parametric perturbations acting on the control system. It is assumed that the perturbations p are constant in time and belong to a known set $p \in W \subset R^k$. The perturbations q are regarded as arbitrary piecewise continuous functions whose values belong to a given set $q(t) \in Q$. In the problem of magnetic stabilization, for example, the parametric perturbations p are errors of initial orientation of the instrumental frame and the perturbations $q(t)$ are the "parasitic torques" arising because of intrinsic magnetization of the satellite.

Equation (5) should be understood as a brief formal representation of the corresponding integral equation, whereas $\xi(t)$ should be understood as the generalized derivative of the normal process $\zeta(t)$ with independent increments ($\xi\,dt = d\zeta$). The control u belongs to the set $\hat{U}$ of stochastic processes that admit representation (5). In particular, this representation is possible [4] if $u = K(t)x$, where $t \in [t_0, t_k]$, $K(t) \in \mathcal{K}$, and $\mathcal{K}$ is the set of functions whose values are $(n \times s)$-matrices with piecewise continuous elements.

The problem

$$J_0^0 = \max_{p \in W,\, q \in Q} \min_{u \in \tilde{U}} J(p,q,u) \tag{6}$$

is further solved for auxiliary system (5) and the extended set $u(\cdot) \in \tilde{U}$ ($U \subset \tilde{U} \subset \hat{U}$) of controls. Here $\tilde{U} = \{u(\cdot) : u = K\tilde{x}\}$ and $K(t)$ is a matrix with bounded piecewise continuous elements. During the process of solving this problem, the optimal counterstrategy p^0, q^0 and the auxiliary stabilization algorithm u^0 are determined. The algorithm u^0, which provides the best accuracy of stabilization, is found from the inner minimization problem.

If complete and accurate data on the deviations x are available and the control resources are unlimited ($\Omega = R^s$), then the auxiliary optimal control u^0 has the form of linear feedback in deviations: $u^0 = K(t)x$.

If only incomplete data are available, then the auxiliary stabilization algorithm is formed with the use of the deviation estimates $\tilde{x}$ produced by the estimation system on the basis of data from measuring sensors. These data are represented by the measurement vector $z = Hx + H_1q + \eta$, where η are the instrument errors of the measuring sensors being described by the white-noise process [2]. It is assumed that the estimation algorithm has the form [2]

$$\begin{aligned} \dot{\tilde{x}} &= A(t,p)\,\tilde{x} + B(t,p)\,u + C(t,p)\,q + \tilde{K}(z - H\tilde{x}) \\ z &= Hx + H_1 q + \eta \end{aligned} \tag{7}$$

where $\tilde{K}(t)$ is the feedback matrix in this algorithm.

The stabilizing control is determined in the form $u = K\tilde{x}$, where $\tilde{x}$ are the deviation estimates obtained by estimation algorithm (7).

By the separation theorem [4], the problem of finding the best accuracy (6) for system (5), (7) may be formulated as follows:

$$J_0^0 = \max_{p,q} J^0(p,q), \quad J^0(p,q) = \min_{K,\tilde{K}} M\left[x(t_k)^\top S x(t_k) + \int_0^{t_k} (x^\top G x + u^\top N u)\, dt\right]$$

Here K and $\tilde{K}$ are the feedback matrices in the problems of optimal stabilization and estimation, respectively.

If $q \equiv 0$, then the inner minimization problem is reduced to that of choosing a linear stochastic controller on the basis of the quadratic criterion with incomplete measurements.

Let us consider the general case $q \not\equiv 0$ and assume that complete data on the deviations x are available. If the control is given in the form of the linear feedback $u = Kx$, then performance criterion (2) can be rewritten as

$$J = \mathrm{Tr}\left[SP_x(t_k) + \int_0^{t_k} (GP_x + K^\top NKP_x)\, dt\right] \tag{8}$$

where P_x is the matrix of second torques ($P_x = M[xx^\top]$) and Tr is the operation of calculating the matrix trace: $\mathrm{Tr}\,\|a_{ij}\| = \sum_{i=1}^{n} a_{ii}$ (the sum of the diagonal elements).

The matrix P_x satisfies the dispersion relation

$$\dot{P}_x = (A + BK)P_x + P_x(A + BK)^\top + Cqe_x^\top + e_x(Cq)^\top + D^\top FD, \quad P_x(0) = P_x^0 \tag{9}$$

The stochastic problem of finding a minimum of functional (2) is reduced to the deterministic extremal problem

$$J^0 = \min_K J$$

under condition (9), where the elements of the matrices K and P_x are regarded as controls and phase coordinates of dimension $\dfrac{n(n+1)}{2}$, respectively.

Note that $P_x = P + e_x e_x^\top$ (where $P = M\left[\mathring{x}\mathring{x}^\top\right]$, $\mathring{x} = x - e_x$, $e_x = M[x(t)]$) and e_x satisfies the equation

$$\dot{e}_x = (A + BK)\, e_x + Cq \tag{10}$$

Then it follows from expression (8) that

$$\begin{aligned} J^0 &= \mathrm{Tr}\left[SP_x(t_k) + \int_0^{t_k} (GP_x + K^\top NKP_x)\, dt\right] = J_1^0 + J_2^0 \\ J_1^0 &= \mathrm{Tr}\left[SP(t_k) + \int_0^{t_k} (GP + K^\top NKP)\, dt\right] \\ J_2^0 &= e_x^\top S e_x + \int_0^{t_k} e_x^\top (G + K^\top NK)\, e_x\, dt \end{aligned} \tag{11}$$

The solution of extremal problem (6) $J_0^0 = J(p^0, q^0)$ is obtained by a method of successive approximations.

The inner minimization problem with respect to the coefficients K should be solved at each step of the iteration procedure. We solve this problem by the method of successive approximations, too. The feedback coefficients are represented by two components as follows: $K = K_x + K_e$. The first component K_x (K_e should be fixed) is obtained as a solution of the problem $\min\limits_{K_x} J_1^0$; therefore, $K_x = -N^{-1}B^\top\mathcal{L}$, where $\mathcal{L}$ is the symmetric matrix satisfying the differential equation

$$\dot{\mathcal{L}} + \mathcal{L}A' + A'^\top\mathcal{L} + G' - \mathcal{L}BN^{-1}B^\top\mathcal{L} = 0, \quad \mathcal{L}(t_k) = S \tag{12}$$

and $A' = A + BK_e$, $G' = G + K_e N K_e^\top$. The funcional J_2^0 can be represented as

$$J_2^0 = e_x^\top S e_x + \int_0^{t_k} (e_x^\top G'' e_x u_e^\top N u_e)\, dt$$

where $u_e = K_e e_x$ and $G'' = G + K_x^\top N K_x$. The coefficients K_e are obtained as a solution of the minimization problem for the quadratic funtcional J_2^0 and linear system (10):

$$\begin{aligned} &J_2^0 = e_x^\top S e_x + \int_0^{t_k} (e_x^\top G'' e_x u_e^\top N u_e)\, dt \to \min_{u_e} \\ &\dot{e}_x = A'' e_x + B u_e + C q \end{aligned} \tag{13}$$

where $u_e = K_e e_x$, $G'' = G + K_x^\top N K_x$, and $A'' = A + BK_x$. After several iterations, we obtain optimal values of the coefficients K as a solution of the inner problem.

Now we turn to the outer problem. If $p \in W$ is fixed, then the problem of finding $\max\limits_{q(\cdot)\in Q} J^0$ is reduced to the extremal problem of control in $q(t)$ in the linear system

$$\dot{e}_x = A(p) e_x + B(p) K e_{\tilde{x}} + C q(t) \tag{14}$$

with the quadratic performance criterion

$$J_2^0 = e_x^\top S e_x + \int_0^{t_k} e_x^\top (G + K^\top N K)\, e_x\, dt \to \max_{q(\cdot)\in Q} \tag{15}$$

Its solution may be obtained, for example, by the Krylov–Chernous'ko method of successive approximations [5] whose convergence is proved in some special cases (see [6]). At the next step (once the perturbations $q(t)$ have been determined), the problem of finding $\max\limits_{p\in W} J^0$ may be considered as a problem of nonlinear programming in parameters $p \in W$, which can be solved by an appropriate numerical method [7].

4. Accuracy testing of the stabilization algorithm. At the second step of the strategy, the stabilization algorithm of forming control signals u^r is tested with the use of the results obtained at the first step. This algorithm may be unknown for the tested system, i.e., the system considers this algorithm as a "black box" with known inputs and outputs.

The testing procedure requires statistical tests for system (1) to be performed and estimates the performance criterion (2) with the use of methods of mathematical statistics. For this purpose, system (1) is integrated with known values of p^0 and $q^0(t)$; to accomplish this, the errors ξ in the system and the readings of the sensors are modeled. The control signals u^r are obtained at the output of the "black box" and system (1) is closed. After processing the results of integration, the statistical mean value of stabilization accuracy of the algorithm $J^0(p^0, q^0(t), u^r)$ is calculated; this value is compared with the best accuracy J_0^0 obtained at the first stage:

$$\kappa = \frac{J^0(u^r)}{J_0^0}$$

This comparison allows us to judge the quality of stabilization by the control signals u^r.

The interval estimates of the functional $J^0(p^0, q^0(t), u^r)$ permit determining the number of necessary statistical tests at the second stage. If the best accuracy J_0^0 falls on the confidence interval in $J^0(p^0, q^0(t), u^r)$, then we should return to the second stage and increase the number of tests.

5. Testing of algorithms for active magnetic stabilization. **1.** Let us consider the application of the above algorithm to the problem of testing the algorithms for active magnetic stabilization of the satellite. The equations

$$\begin{aligned} A\ddot{\alpha}_1 + \omega_0\,(C + B - A)\,\dot{\alpha}_3 + 4\omega_0^2\,(B - C)\,\alpha_1 &= M_1^a + M_1^b + \Delta M_1^g \\ B\ddot{\alpha}_2 + 3\omega_0^2\,(A - C)\,\alpha_2 &= M_2^a + M_2^b + \Delta M_2^g \\ C\ddot{\alpha}_3 + \omega_0\,(B - A - C)\,\dot{\alpha}_1 + \omega_0^2\,(B - A)\,\alpha_3 &= M_3^a + M_3^b + \Delta M_3^g \end{aligned} \tag{16}$$

describe small oscillations about the center of mass in projections on the axes of the coupled system. Here A, B, and C are the torques of inertia; $\omega_0(t)$ is the angular velocity of orbital motion; $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is the vector of small rotation of the coupled system $Mx_1x_2x_3$ relative to the orbital system $My_1y_2y_3$.

Let us introduce the following notations: $\delta = \{\delta_{ij}\}$ is the matrix of rotation of the axes $Mz_1z_2z_3$ of the instrumental frame relative to the coupled system Mx; $u = (u_1, u_2, u_3)$ are the projections of the active torque on the axes of the system Mz (this torque is generated by current coils); (I_1^n, I_2^n, I_3^n) are the projections of the intrinsic magnetic torque arising because of magnetization of the satellite; $T = (T_1, T_2, T_3)$ are the projections of the Earth's magnetic strength vector on the axes of the orbital system My; ΔM_1^g, ΔM_2^g, and ΔM_3^g are the projections of the disturbance torques (for which a statistical description is available only) on the axes of the coupled system (for example, these torques may be represented by the truncated Fourier series

$$\Delta M_i^g = d_{1i} + d_{2i}\cos\omega_0 t + d_{3i}\sin\omega_0 t + d_{4i}\cos 2\omega_0 t + d_{5i}\sin 2\omega_0 t + d_{6i}, \quad i = 1, 2, 3 \tag{17}$$

where d_{1i}, d_{2i}, d_{3i}, d_{4i}, $d_{15} = \text{const}$ and d_{6i} is the high-frequency component of the white-noise type).

The projections of the active magnetic torque on the axes of the coupled system are denoted by $u_x = [E - \hat{\delta}]\, u_z$. The projections of the control torque $M^a = u \times T$ are written down in the form

$$\begin{aligned} M_1^a &= u_2(T_3 - T_2\delta_1) + u_1(T_2\delta_2 + T_3\delta_3) - u_3(T_2 + T_3\delta_1) + \sum_{i,j}\alpha_i u_j \lambda_{ij1}(\delta) \\ M_2^a &= u_3(T_1 - T_3\delta_2) + u_2(T_1\delta_1 + T_3\delta_3) - u_1(T_3 + T_1\delta_2) + \sum_{i,j}\alpha_i u_j \lambda_{ij2}(\delta) \\ M_3^a &= u_2(T_3 - T - 2\delta_1) + u_1(T_2\delta_2 + T_3\delta_3) - u_3(T_2 + T_3\delta_1) + \sum_{i,j}\alpha_i u_j \lambda_{ij3}(\delta) \end{aligned} \tag{18}$$

Here $\lambda_{ijk}(\delta)$ are the coefficients at the nonlinear components of the active control torque (18). Similar expressions can be written for the components of the disturbance magnetic torque.

The measurement vector may be represented in the form $z = T_z^m - T^*$, where T_z^m is the vector composed of the measured components of the field strength at the point M and T^* is the information on projections of the Earth's magnetic field obtained aboard the satellite. The equality $T^* = T_y$ is valid when the center-of-mass position M of the satellite is known with certainty. The representation

$$T_z^m = [E + \Delta]\left[E + \hat{\delta}'\right]\left[E + \hat{\delta}\right][E + \hat{\alpha}]\, T_y + \xi_0 + \xi_1 \sin\omega_0 t + \xi_2\cos\omega_0 t + \xi_3 \tag{19}$$

holds up to small quantities of second order. Here $\xi_0 = (\xi_1^0, \xi_2^0, \xi_3^0)$ is the systematic error of measurement (zero drift of the sensors); ξ_1 and ξ_2 are the time-dependent components of the systematic error; ξ_3 are the high-frequency error components represented by white noise; δ' is the vector of small rotation of the sensors' sensitivity axes relative to the system Mz; Δ is the matrix of relative measurement errors. The errors in the measurements at the scale of magnetometers are represented by the diagonal elements $(\Delta_{11}, \Delta_{22}, \Delta_{33})$, whereas the off-diagonal elements (Δ_{ij}) are the coefficients of cross sensibility of the magnetic sensor (these coefficients arise because of nonorthogonality of the measurement axes along which the magnetic sensors are located).

If we use the projections on the axes of the system Mx, then relations (19) are written down in the form

$$\begin{aligned} z_k = \Delta_{kk}T_1 + (\alpha \times T)_k + \sum_i \delta_i \kappa_{ik}^2(T_y) + \sum_i \alpha_i \kappa_{ik}^3(T_y) + \sum_{i,j}\alpha_i T_j \kappa_{ijk}^4(\Delta_{ij}, \delta, \delta') \\ + \xi_k^0 + \xi_k^1 \sin\omega_0 t + \xi_k^2 \cos\omega_0 t + \xi_k^3 \end{aligned} \tag{20}$$

The additional equations

$$\begin{cases} \dot{d}_{2i} = d_{3i}, \\ \dot{d}_{3i} = \omega_0^2 d_{2i}, \end{cases} \qquad \begin{cases} \dot{d}_{4i} = d_{5i}, \\ \dot{d}_{5i} = 4\omega_0^2 d_{4i}, \end{cases} \qquad \begin{cases} \dot{\xi}_i^1 = \xi_i^2 \\ \dot{\xi}_i^2 = \omega_0^2 \xi_i^1 \end{cases}$$

$$\dot{d}_{1i} = 0, \quad \dot{\xi}_i^0 = 0, \quad \dot{\kappa}_i = 0, \quad i = 1, 2, 3$$

used to describe the constants in expressions (17) and in the measurement vector (20) are supplemented to equations (18). Assuming that α_i and δ_j are small, we ignore the nonlinear terms in (18) and (20). Then, the equations of motion of the control system may be represented in the form

$$\begin{aligned} \dot{x} &= A(t,p)\, x + B(t,p)\, u + C(t,p)\, q + D(t)\, \xi \\ \dot{\tilde{x}} &= A(t,p)\, \tilde{x} + B(t,p)\, u + C(t,p)\, q + \tilde{K}(z - H\tilde{x}) \\ z &= H(t,p)\, x + H^1(t)\, q + \eta \end{aligned}$$

Here $x = (\alpha_1, \alpha_2, \alpha_3, \dot\alpha_1, \dot\alpha_2, \dot\alpha_3, d_{1i}, d_{2i}, d_{3i}, d_{4i}, d_{5i})$, $i = 1, 2, 3$; $u = (u_1, u_2, u_3)$; $p = (\delta_1, \delta_2, \delta_3)$; $q = (I_1^n, I_2^n, I_3^n)$; $\xi = (d_{61}, d_{62}, d_{63}, \xi_1^3, \xi_2^3, \xi_3^3)$.

This system is similar to equations (5) and (7); therefore, the strategy of testing (i.e., obtaining the best accuracy and forming the optimal counterstrategy (p^0, q^0)) can be found solving the extremal problem (6).

2. Let us consider in more detail the calculation of the following three characteristics: a value of the game, the optimal counterstrategy, and an estimate of stabilization quality. For doing this, we consider the stabilization of pitching oscillations of the satellite moving along the circular polar orbit.

It is assumed for simplicity that the Earth's magnetic field strength is formed by the field T specified by the dipole and the fluctuating component ΔT. Let the satellite move uniformly along the circular orbit. The orbital system of coordinates is denoted by $Oy_1y_2y_3$. The magnetic torque M^a generated by the current coil is represented in the form $M^a = M_u^0 * u$, where $u = \{1, 0, -1\}$ is the control.

The coordinate system associated with the principle axes of inertia of the satellite is denoted by Ox. Let α be the angle of rotation of the system Ox relative to the orbital system Oy. The system Oz is associated with the instrumental frame of the satellite. This system is rotated relative to Ox through the angle $\delta_0 + \delta$, where δ_0 is the required value for the angle of rotation and δ is the error of realization of the instrumental frame. The equation for pitching oscillations may be written down in the form

$$B\ddot\alpha + 3\omega_0^2(A - C)\sin\alpha\cos\alpha = M^a + M^b + \Delta M_\xi$$

Here A, B, and C are the inertia torques of the satellite; M^a and M^b are the control and disturbance magnetic torques; ΔM_ξ are the disturbance torques of external forces. Let $\Delta T_y = (\tilde\Delta_1, 0, \tilde\Delta_3)$ be the projections of perturbations of the Earth's magnetic field strength on the axes of the system Oy. The intrinsic magnetization of the satellite causes the magnetic torque whose projections on the axes of the coupled system Ox are $I_\sigma = (n_1, 0, n_3)$. It is assumed that a single current coil (whose magnetic torque is oriented along the z_3-axis) is mounted on the satellite. Then, the control torque is

$$\begin{aligned} M^a = 0.5M_u^0 u \Big[& 2e_0\tilde\Delta_1\cos(\alpha + \delta_0 + \delta) - 2e_0\tilde\Delta_3\sin(\alpha + \delta_0 + \delta) \\ &- B_0\big(3\sin(\alpha + \delta_0 + \delta - \tau) + \sin(\alpha + \delta_0 + \delta + \tau)\big)\Big] \end{aligned}$$

where $B_0 = \mu_0 H_0 / r_0^3$, H_0 is the magnetic torque of the Earth's dipole, μ_0 is the coefficient of magnetic permittivity, and r_0 is the orbit radius of the satellite.

The disturbance torque is

$$M^b = B_0 M_u^0 \left[\sigma_3(\Delta_1 + \sin\tau) + \sigma_1(-\Delta_3 - 2\cos\tau) + \alpha\left(\sigma_1(-\Delta_1 - \sin\tau) + \sigma_3(-\Delta_3 - 2\cos\tau)\right)\right]$$

If we retain only the terms linear in α, then the oscillatory equations can be written in the dimensionless form as follows:

$$\begin{aligned} &\ddot\alpha + \left(b^2 - b_{50}\sigma_1 - b_{60}\sigma_3 + b_{20}u\right)\alpha = u[b_{00} + b_{21}\delta] + \sigma_1 b_{31} + \sigma_3 b_{41} + \xi_1 \\ &\quad + \Delta_1[u(b_{70}u + b_{71}(\delta + \alpha)) + b_{40}\sigma_3] + \Delta_3[u(b_{80}u + b_{81}(\delta + \alpha)) + b_{80}\sigma_1] \end{aligned} \tag{21}$$

Here $\tau = \omega_0 * t$, $M_0 = \dfrac{M_u^0 B_0}{\omega_0^2 B}$, $\Delta_i = \dfrac{e_0\tilde\Delta_i}{B_0}$, $\sigma_i = \dfrac{n_i}{M_u^0}$ are the dimensionless variables, and

$$b^2 = 3\frac{A - C}{B}, \quad \xi_1 = \frac{\Delta M_\xi}{B\omega_0^2}, \quad b_{10} = 1, \quad b_{30} = -M_0, \quad b_{40} = M_0$$

$$b_{00} = M_0\left(-\frac{3}{2}\sin(\delta_0 - \tau) + \frac{1}{2}\sin(\delta_0 + \tau)\right), \quad b_{20} = M_0\left(\frac{3}{2}\cos(\delta_0 - \tau) + \frac{1}{2}\cos(\delta_0 + \tau)\right)$$

$$b_{30} = -1, \quad b_{40} = 1, \quad b_{21} = -b_{20}$$

$$b_{50} = M_0\sin\tau, \quad b_{60} = 2M_0\cos\tau, \quad b_{70} = M_0\cos\delta_0, \quad b_{80} = -M_0\sin\delta_0$$

$$b_{31} = -2M_0\cos\tau, \quad b_{41} = M_0\sin\tau, \quad b_{71} = -M_0\sin\delta_0, \quad b_{81} = -M_0\cos\delta_0$$

The Mexican satellite "Satex" travelling along the orbit of radius $r_0 = 800$ km has the active magnetic torque $M_u^0 = 4\,A \cdot \mathrm{m}^2$; for this case, the parameters M_0 and b^2 are equal to 1.519 and 2.936, respectively.

Let $p = \delta$, $q_1 = \sigma_1$, $q_2 = \sigma_3$, $\xi_1 = \dfrac{\Delta M_\xi}{B\omega_0^2}$, $\xi_2 = \Delta_1$, $\xi_3 = \Delta_3$, $x = (\alpha, \dot\alpha)$, $q = (q_1, q_2)$, $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$. Equation (21) can be rewritten up to $\alpha\Delta_i$, $\delta\Delta_i$, $\alpha\sigma_i\Delta_i$ as follows:

$$\dot x = A(t)x + B(t, p)u + C(t)q + D(t)\xi \tag{22}$$

Here

$$A = \begin{pmatrix} 1 & 1 \\ -b^2 & 0 \end{pmatrix}, \quad B(p) = \begin{pmatrix} 0 \\ b_{00} + b_{20}p \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ b_{31} & b_{41} \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ b_{10} & b_{20} & b_{70} & b_{80} \end{pmatrix}, \quad M\left[\xi_4(t)\,\xi_4(t')\right] = D_0 e^{-\gamma t}\delta(t-t')$$

The parameters D_0 and γ are chosen in such a way that the solution of auxiliary system (22) on the segment $[0, t_k]$ differs little from the solution of nonlinear system (17).

It is assumed that the error $p \in W$ of initial orientation of the instrumental frame belongs to a known segment and the components of the vector $q(t)$ are arbitrary time functions whose values are known up to the set $q(t) \in Q$. The components of the vector ξ are regarded as independent white noise with intensities $\tilde{F}_i$.

At the first stage of testing, problem (6) of finding the best accuracy is solved by a successive approximation method. It is assumed for simplicity that complete data on the state vector x are known.

The worst perturbations $q^0(t)$ are obtained by solving the linear problem (14) with the quadratic performance criterion (15).

The dependence of the performance criterion $J^0(p, q^0)$ on p is presented in Figure 2 for the following initial data:

$$t_k = 12; \quad \tilde{F}(1,1) = 0.1; \quad \tilde{F}(2,2) = 0.5;$$

$$\delta_0 = 0.7; \quad |q_1(t)| \leqslant 0.1; \quad |q_2(t)| \leqslant 0.1$$

$$N = 1; \quad S = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}; \quad G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

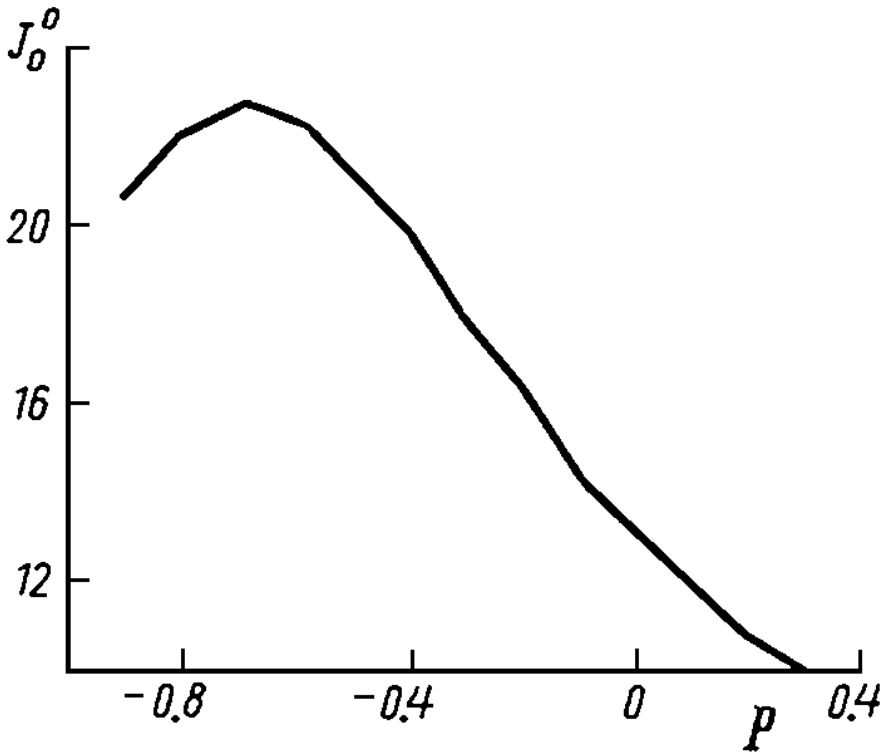

Fig. 2

The worst perturbation $p^0 = -0.7$ corresponds to the case when the current coil is located in such a manner that its magnetic torque is directed along the x_3-axis. It follows from Figure 2 that the position when the frame is perpendicular to the x_3-axis provides the best stabilization of pitching oscillations.

The behavior of mean value of the satellite's rotation angle e_{x1} and its angular velocity e_{x2} as well as the worst perturbation $q_1^0(t)$ for the worst $p^0 = -0.7$ are shown in Figure 3.

At the second stage of testing, we perform statistical tests for the stabilization of oscillations with the use of the control signals $u^r = -\operatorname{sign}\left[(b_{00} + b_{12}p)\,\dot{\alpha}\right]$ (this control is examined in [8]).

As for the sample mean value of performance criterion (2) and its terminal part corresponding to the control u^r, the dependence of this value on the test number k is represented in Figure 4. As is seen from this figure, in order to obtain a stable statistical estimate of the functional, we should carry out approximately 60–80 tests. Statistical tests of the control algorithm $u^0 = K(p^0)x$ obtained at the first stage were also performed.

The time dependence of elements of the covariance matrix of the state vector P_{ij}^0 for the optimal control u^0 (the thick lines) and the covariance values P_{ij}^r for the control u^r to be tested (the thin lines) are represented in Figure 5.

The results of testing are given in Table 1. The sample mean values m_J, the sample variances σ_J, and the corresponding confidence intervals I_m, I_σ (with the confidence probability $p_\alpha = 0.95$) are shown in the first row for the best accuracy J_0^0 of criterion (2) (the control u^0). The second row contains the results of statistical tests for the quality functional J_r^0 corresponding to the control u^r to be tested. The last two rows of the table contain similar values for the terminal part of functional (2).

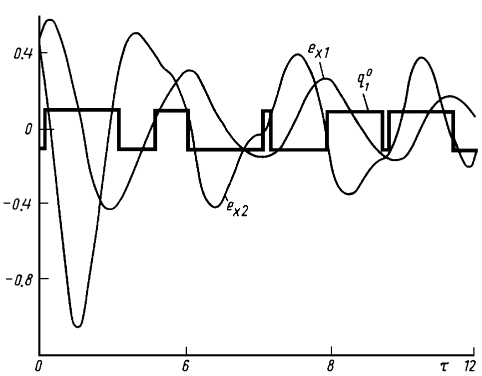

Fig. 3

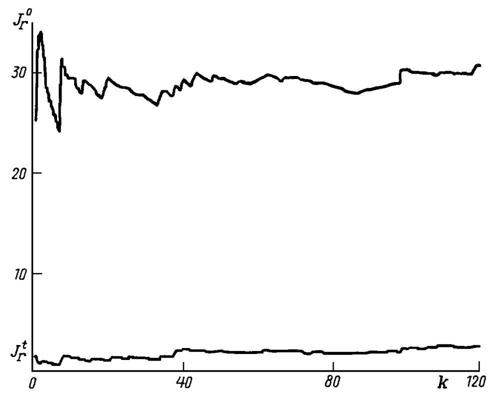

Fig. 4

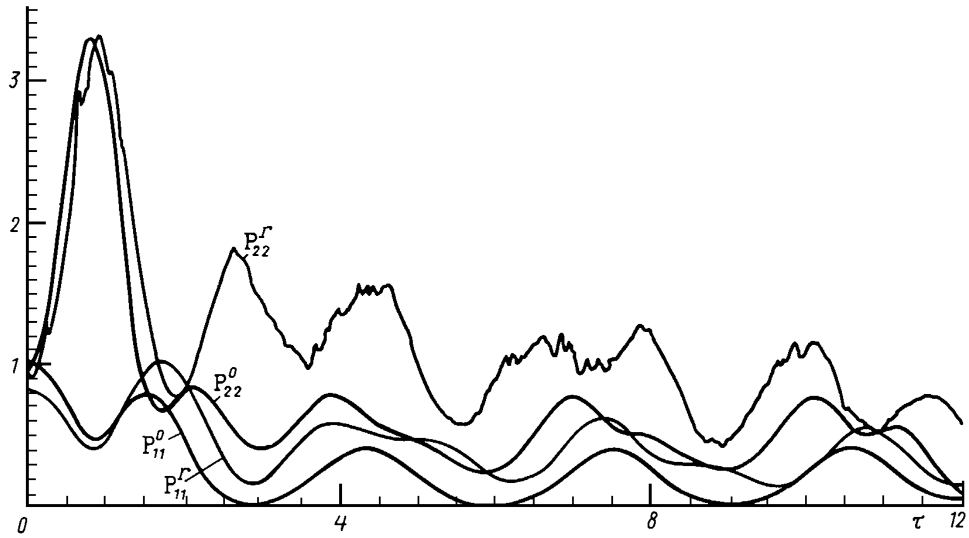

Fig. 5

Table 1

J	m_J	σ_J	I_m (0.05)	I_σ (0.05)
J_0^0	19.8	1.037	19.63–20.01	0.92–1.188
J_r^0	29.84	2.05	29.47–30.21	1.82–2.348
J_0^t	0357	0.092	0.340–0.374	0.85–1.102
J_r^t	3.27	0.96	3.09–3.44	0.85–1.102

Table 2

J	m_J	σ_J	I_m (0.05)	I_σ (0.05)
J_0^0	40.55	2.22	40.15–40.96	1.97–2.55
J_r^0	66.06	13.05	63.7–68.42	11.58–14.94
J_0^t	2.28	0.35	2.21–2.35	0.316–0.407
J_r^t	3.01	1.48	2.75–3.28	1.313–1.695

It follows from Table 1 that in terms of the chosen criterion the quality of the control u^r is $\kappa = \dfrac{29.84}{19.8} = 1.507$ times worse than it is for the "best" result of control; with a probability of 0.95, κ belongs to the range $1.472 < \kappa < 1.538$.

If we consider only the terminal part of criterion (2), then the results are essentially worse: $\kappa = 9.3$. This is due to the fact that the control u^0 responds to small deviations of the satellite from its stationary position better than the discrete control u^r. If stronger requirements are imposed on the auxiliary control u^0, then the results of testing become better. As an example, the results of statistical tests performed with $N = 3.5$ in functional (2) are given in Table 2.

As is seen from Table 2, with this performance criterion the result of testing the terminal part of functional (2) is $\kappa = \dfrac{3.01}{2.28} = 1.32$ if $N = 3.5$.

The result of testing for functional (2) ($\kappa = \dfrac{66.06}{40.55} = 1.628$) is no better than in the case of the criterion for $N = 1$.

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